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Origin of pure spinor superstring

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Abstract: The pure spinor formalism for the superstring, initiated by N. Berkovits, is derived at the fully quantum level starting from a fundamental reparametrization invariant and super-Poincaré invariant worldsheet action. It is a simple extension of the Green-Schwarz action with *doubled* spinor degrees of freedom with a compensating *local supersymmetry* on top of the conventional κ-symmetry. Equivalence to the Green-Schwarz formalism is manifest from the outset. The use of free fields in the pure spinor formalism is justified from the first principle. The basic idea works also for the superparticle in 11 dimensions.

Keywords: BRST Symmetry, Space-Time Symmetries, Superstrings and Heterotic Strings.

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Contents

1. Introduction

The pure spinor (PS) formalism, initiated by Berkovits [1], is a remarkable construct for describing the superstring. Essentially everything is controlled by the nilpotent BRST-like operator $Q = \int dz \lambda^\alpha d_\alpha$, where $d_\alpha = p_\alpha + i\partial x^m (\gamma_m\theta)_\alpha + \frac{1}{2}(\gamma^m\theta)_\alpha(\theta\gamma_m\partial\theta)$ is the spinor covariant derivative and λ^α is a bosonic chiral spinor satisfying the pure spinor constraints $\lambda^\alpha\gamma^m_{\alpha\beta}\lambda^\beta = 0$. All the basic fields are postulated to be free and constitute a conformal field theory with vanishing central charge. The cohomology of Q correctly describes the lightcone spectrum of the superstring [2] and appropriate vertex operators and a set of rules can be given to compute the scattering amplitudes in a super-Poincaré covariant manner [1], even to all loops in principle [3]. Moreover, it can be coupled to backgrounds including Ramond-Ramond fields in a quantizable and covariant way [1, 4], in distinction to the conventional Ramond-Neveu-Schwarz (RNS) [5] and Green-Schwarz (GS) [6] formalisms, where one meets difficulties. This feature makes the PS formalism particularly promising for deeper understanding of the gauge/string correspondence [7]. For many other developments, the reader is referred to [8] as well as a review [9].

Successful as it has been, there are a number of fundamental questions to be clarified on the PS formalism. One of them is the understanding of the quantization, especially that of the pure spinor λ^α. Even though the free-field postulate is powerful and attractive, the ghost-like field λ^α, subject to the quadratic constraints, is not truly free. It is hard to imagine how such a field could arise naturally in a conventional quantization process. Also, solving the constraints breaks manifest Lorentz covariance in the intermediate steps of computations. This prompted an attempt for a fully covariant formulation in an extended Hilbert space where the pure spinor constraints are removed [10]. Subsequently an alternative formulation without pure spinor constraints, which is more closely related to the original PS formalism, was proposed [11 − 13]. These proposals added interesting insights and demonstrated certain advantage of the enlarged field space, but they are yet to be fully developed. Another basic question is the origin of the BRST-like operator Q. Since the components of d_α do not form a closed first class algebra, Q cannot immediately be understood as a conventional BRST charge.

Clearly, all these and other related questions have their roots in the lack of our knowledge of the fundamental action and its underlying symmetries for the PS formalism. There exist several thought-provoking attempts [14, 15] to derive the PS formalism but their success has been partial: In [14] the BRST charge Q was still postulated, not derived, and the Q-invariant modification of the GS action contained Lorentz non-invariant expressions. On the other hand, the analysis of [15], performed at the classical level, identified Q as one of the supercharges of the twisted version of the $N = 2$ superconformal algebra, which was different from the geuine BRST charge that followed from the proposed action.

However, during the past year, some concrete hints have been obtained which indicate that, as had been suspected, PS formalism is intimately related to the GS formalism. First, in the work [13] proving the equivalence of the lightcone BRST treatment of the GS formalism to the extended version of the PS formalism, it was noted that the usual pair of reparametrization ghosts, commonly denoted by (b, c), can be identified with one of the five pairs of ghosts which compensate for the removal of the PS constraints. This basic structure reappeared in a more recent work [16], where further important hints were provided. Introducing a conjugate pair of free fields $(\theta^\alpha, p_\alpha)$ into the GS formalism in the semi-lightcone gauge [17] described by the 8-component self-conjugate SO(8) chiral spinor S_a, the authors ingeniously constructed a set of 17 operators, denoted in [16] by $\hat{d}_a, \hat{d}_{\dot{a}}$ and $\tilde{T}$, which form a closed first class algebra. This algebra, although not derived systematically, allows one to gauge away the newly added pair $(\theta^\alpha, p_\alpha)$ thereby ensuring the correct degrees of freedom. The corresponding BRST operator $\hat{Q}$ is readily constructed by introducing 16 unconstrained bosonic spinor ghosts $(\hat{\lambda}^a, \hat{\lambda}^{\dot{a}})$ and a pair of fermionic ghosts (b, c). Then it was shown that the cohomology of $\hat{Q}$ is the same as that of Q of the PS formalism with the PS constraints. In this mechanism, the fields (S_a, b, c) play exactly the same role as the 5 pairs of ghosts $(b_P, c_P)_{P=1\sim5}$ in [11, 13].

Stimulated by these developments, in particular the idea in [16] that an extra local fermionic symmetry can lead naturally to the BRST charge in an extended space similar to the one in our formalism [11 − 13], we attempted to realize this type of local fermionic symmetry from the very beginning in a completely covariant fashion.

The result of our investigation, to be described in this paper, is a worldsheet action with the reparametrization and the super-Poincaré invariance, from which one can *derive* the pure spinor formalism from the first principle at the fully quantum level: Although the action is non-linear just as in the GS formalism, through a conventional Dirac analysis of its local symmetries, the first class algebra devised in [16] emerges naturally and systematically, including the rules that the fields involved obey free-field operator products.

JHEP05(2005)046

In section 2, we shall describe our fundamental action and its symmetries. The action is a simple extension of the Green-Schwarz action with *doubled* spinor degrees of freedom, θ and $\tilde{\theta}$, with a compensating *local supersymmetry* on top of the usual κ-symmetry [18]. If one gauge-fixes θ to be zero by this extra local symmetry, one immediately recovers the conventional GS formalism in terms of $\tilde{\theta}$. In section 3, we will perform the Dirac analysis of the constraints generated by our action. After separating out the first and the second class constraints, we will impose the semi-lightcone gauge for $\tilde{\theta}$, without spoiling the local supersymmetry and conformal symmetry. The Lorenz covariance is necessarily broken for terms involving $\tilde{\theta}$ but not in the sector consisting of θ alone. This procedure leads to a closed algebra of first class constraints under the Dirac brackets. At this stage, we will encounter a grave problem that the Dirac brackets between the basic variables are no longer canonical, acquiring non-linear modifications. Fortunately, we are able to show in section 4 that there is a graceful way out of this apparent impasse: We discover that there exists a set of field-redefinitions such that the new basic fields become completely free under the Dirac bracket. Quantization, which is now essentially trivial, will be performed in section 5. It is straightforward to find the quantum modifications to the constraint operators so that they continue to form a closed first class algebra. Remarkably, this quantum algebra will be seen to be identical to the one engineered in [16]. The advantage of our formulation is that the highly non-trivial form of the generators forming this algebra follows systematically from the underlying action without any guess work. The rest of the the procedure to reach the PS formalism was already fully described in [16]. For the convenience of the reader, however, we shall briefly reproduce the essence of the argument in a slightly more streamlined fashion. This completes the derivation of the PS formalism from our fundamental action. As an application of our basic idea, we will briefly demonstrate in section 6 that our formalism works straightforwardly for the superparticle in 11 dimensions as well. Starting from a covariant Brink-Schwarz [19] like action one can derive the PS formalism, which coincides with the one introduced in [20], capable of describing the 11 dimensional supergravity in a covariant way. In section 7 we briefly summarize our results and indicate some interesting directions for further research.

2. Action and its symmetries

The basic fields of our theory are the string coordinate x^m and two types of Majorana-Weyl spinors, $\theta^{A\alpha}$ and $\tilde{\theta}^{A\alpha}$, of the same chirality.[1] The vector index m runs from 0 to 9, the spinor index α runs from 1 to 16 and the index $A = 1, 2$ labels the two sets within each type. We will often distinguish them by unhatted and hatted notations, such as $\theta^\alpha \equiv \theta^{1\alpha}$, $\hat{\theta}^\alpha \equiv \theta^{2\alpha}$, etc. They will eventually become left (holomorphic) and right (anti-holomorphic) variables. The worldsheet coordinate will be denoted by $\xi^i = (t, \sigma)$, $i = 0, 1$. As for the γ-matrices, we use 16-dimensional γ^m, which are real and symmetric. We will employ left derivatives throughout.

[1]We will describe the type IIB case in this paper. The type IIA case is similar.

Our fundamental action is given by[2]

$$
\begin{aligned}
S &= \int d^2\xi(\mathcal{L}_K + \mathcal{L}_{WZ})\,, \\
\mathcal{L}_K &= -\frac{1}{2}\sqrt{-g}\,g^{ij}\Pi_i^m\Pi_{mj}\,, \\
\mathcal{L}_{WZ} &= \epsilon^{ij}\Pi_i^m(W_{mj}-\hat{W}_{mj}) - \epsilon^{ij}W_i^m\hat{W}_{mj}\,,
\end{aligned} \tag{2.1}
$$

where

$$
\begin{aligned}
\Pi_i^m &\equiv \partial_i x^m - \sum_A i\partial_i(\theta^A\gamma^m\tilde{\theta}^A) - \sum_A W_i^{Am}\,, \\
W_i^{Am} &\equiv i\Theta^A\gamma^m\partial_i\Theta^A\,, \qquad \Theta^A \equiv \tilde{\theta}^A - \theta^A\,.
\end{aligned} \tag{2.2}
$$

It can be obtained from the conventional Green-Schwarz action in terms of a spinor field Θ^A by the following simple procedure. Namely, we double the spinor degrees of freedom by setting $\Theta^A = \tilde{\theta}^A - \theta^A$, where $\tilde{\theta}^A$ and θ^A are independent fields, and add interaction terms $\sum_A i\partial_i(\theta^A\gamma^m\tilde{\theta}^A)$ in Π_i^m to realize important symmetries described below.

Besides the manifest reparametrization and super-Poincaré invariance, this action possesses the following three types of fermionic symmetries. First the global spacetime supersymmetry is realized as

$$
\delta\theta^A = \epsilon^A\,, \qquad \delta\tilde{\theta}^A = 0\,, \quad (\delta\Theta^A = -\epsilon^A)\,, \tag{2.3}
$$

$$
\delta x^m = \sum_A i\epsilon^A\gamma^m\theta^A\,. \tag{2.4}
$$

Under this transformation Π_i^m, and hence $\mathcal{L}_K$, is invariant. The Wess-Zumino part $\mathcal{L}_{WZ}$ is also invariant since the transformation for Θ^A is just as in the usual GS case. Secondly there is a *local supersymmetry* defined by

$$
\delta\theta^A = \chi^A\,, \quad \delta\tilde{\theta}^A = \chi^A\,, \quad (\delta\Theta^A = 0)\,, \tag{2.5}
$$

$$
\delta x^m = \sum_A i\chi^A\gamma^m\Theta^A\,, \tag{2.6}
$$

where $\chi^A(\xi)$ is a local fermionic parameter. Since Θ^A is invariant, so is W_i^{Am}. It is easy to check that Π_i^m is also invariant. Note that by using this symmetry one can gauge-fix θ^A to zero, upon which the action reduces precisely to the conventional GS action for $\tilde{\theta}^A$. Therefore the equivalence of our theory to the GS formalism is obvious from the outset. On the other hand, as we shall see, keeping this new local symmetry till the end will lead us naturally, though non-trivially, to the PS formalism. Finally, the third fermionic symmetry present is the local κ-symmetry, to be described shortly.

3. Analysis of constraints

In this paper, we shall analyze and quantize this system in the hamiltonian formulation. The path-integral quantization will be addressed in a separate work. The canonical hamiltonian density is most efficiently obtained by employing the ADM parametrization of the

[2]For simplicity, we set the string tension to unity until we come to the quantization.

JHEP05(2005)046

worldsheet metric. Namely, we parametrize the metric as $ds^2 = -(Ndt)^2 + \gamma(d\sigma + N^1 dt)^2$, where N and N^1 are the lapse and the shift functions and γ is the spatial part of the metric. Since the procedure is completely standard, we only record the relevant results. The definitions of the momenta k^A_α and $\tilde{k}^A_\alpha$ conjugate to $\theta^{A\alpha}$ and $\tilde{\theta}^{A\alpha}$ respectively produce the fermionic constraints of the form

$$\begin{aligned} D^A_\alpha &= k^A_\alpha + i(\not{k}\tilde{\theta}^A)_\alpha + i\left(k^m + \eta_A(\Pi^m_1 - W^{Am}_1)\right)(\gamma_m\Theta^A)_\alpha = 0\,, \\ \tilde{D}^A_\alpha &= \tilde{k}^A_\alpha - i(\not{k}\theta^A)_\alpha - i(k^m + \eta_A(\Pi^m_1 - W^{Am}_1))(\gamma_m\Theta^A)_\alpha = 0\,, \end{aligned} \qquad (3.1)$$

where $k^m = \partial\mathcal{L}/\partial\dot{x}_m$ is the momentum conjugate to x_m, $\not{k} \equiv k^m\gamma_m$, and $\eta_1 = -\eta_2 = 1$. Note that their sum, which generates the local supersymmetry alluded to above, takes a very simple form. The hamiltonian density is then given, up to the above constraints, by

$$\mathcal{H} = \frac{N}{\sqrt{\gamma}}T_0 + N^1 T_1\,, \qquad (3.2)$$

where $T_0 = \frac{1}{2}((k - W_1 + \hat{W}_1)^2 + \Pi_1^2)$ and $T_1 = (k - W_1 + \hat{W}_1)\cdot\Pi_1$, with the notations $A^2 \equiv A^m A_m\,, A\cdot B \equiv A^m B_m$. Demanding consistency with the vanishing of the momenta conjugate to N and N^1, we get the constraints $T_0 = T_1 = 0$. More convenient combinations are

$$T_+ \equiv \frac{1}{2}(T_0 + T_1) = \frac{1}{4}\Pi^m\Pi_m\,, \qquad T_- \equiv \frac{1}{2}(T_0 - T_1) = \frac{1}{4}\hat{\Pi}^m\hat{\Pi}_m\,, \qquad (3.3)$$

where Π^m and $\hat{\Pi}^m$ are defined by

$$\Pi^m \equiv k^m - W^m_1 + \hat{W}^m_1 + \Pi^m_1 = k^m + \partial_\sigma x^m - \sum_A i\partial_\sigma(\theta^A\gamma^m\tilde{\theta}^A) - 2W^m_1\,, \qquad (3.4)$$

$$\hat{\Pi}^m \equiv k^m - W^m_1 + \hat{W}^m_1 - \Pi^m_1 = k^m - \partial_\sigma x^m + \sum_A i\partial_\sigma(\theta^A\gamma^m\tilde{\theta}^A) + 2\hat{W}^m_1\,. \qquad (3.5)$$

$T_\pm$ will be identified as parts of the conformal generators. Since the coefficients $N/\sqrt{\gamma}$ and N^1 can now be taken arbitrary, we shall choose $N/\sqrt{\gamma} = 1$, $N^1 = 0$, namely the conformal gauge.

The next step is to perform the complete analysis of constraints a là Dirac. The Poisson brackets for the fundamental fields are taken as[3]

$$\{x^m(\sigma), k^n(\sigma')\}_P = \eta^{mn}\delta(\sigma - \sigma')\,, \quad \{\tilde{\theta}^{A\alpha}(\sigma), \tilde{k}^B_\beta(\sigma')\}_P = -\delta^{AB}\delta^\alpha_\beta\delta(\sigma - \sigma')\,, \qquad (3.6)$$

$$\{\theta^{A\alpha}(\sigma), k^B_\beta(\sigma')\}_P = -\delta^{AB}\delta^\alpha_\beta\delta(\sigma - \sigma')\,, \quad \text{rest} = 0\,. \qquad (3.7)$$

In spite of the fact that the basic quantities such as Π^m_i and $T_\pm$ contain both unhatted and hatted variables, the algebra of constraints turns out to neatly separate into the "left" and the "right" sectors. So, to simplify the description, we shall hereafter concentrate on the "left" sector and comment on the other sector as it becomes necessary.

[3]Once we choose the sign convention for $\left\{x^m, k^n\right\}_P$, the minus sign for $\left\{\theta^\alpha, k_\beta\right\}_P$ is required for various consistency of the theory, such as the closure of the constraint algebra, implementation of the global supersymmetry, etc.

JHEP05(2005)046

In this sector, the basic Poisson brackets among D_α and $\tilde{D}_\alpha$ are

$$
\begin{aligned}
\{D_\alpha(\sigma),\, D_\beta(\sigma')\}_P &= 2i\gamma^m_{\alpha\beta}\Pi_m\delta(\sigma-\sigma')\,, &(3.8)\\
\{\tilde{D}_\alpha(\sigma),\, \tilde{D}_\beta(\sigma')\}_P &= 2i\gamma^m_{\alpha\beta}\Pi_m\delta(\sigma-\sigma')\,, &(3.9)\\
\{D_\alpha(\sigma),\, \tilde{D}_\beta(\sigma')\}_P &= -2i\gamma^m_{\alpha\beta}\Pi_m\delta(\sigma-\sigma')\,. &(3.10)
\end{aligned}
$$

Note that the local supersymmetry generator

$$\Delta_\alpha \equiv D_\alpha + \tilde{D}_\alpha \tag{3.11}$$

has vanishing Poisson bracket with any linear combinations of D_α and $\tilde{D}_\alpha$, including itself. From now on, we will take Δ_α and $\tilde{D}_\alpha$ as the basic fermionic constraints.

As for T_+, it satisfies the Virasoro algebra of the form $\{T_+(\sigma),\, T_+(\sigma')\}_P = 2T_+(\sigma)\delta'(\sigma-\sigma') + \partial_\sigma T_+(\sigma)\delta(\sigma-\sigma')$, where $\delta'(\sigma-\sigma') \equiv \partial_\sigma\delta(\sigma-\sigma')$. In fact another weakly vanishing quantity $t_+ \equiv \partial_\sigma\Theta^\alpha\tilde{D}_\alpha$, which commutes with T_+, forms the same Virasoro algebra. Their sum will serve as the total Virasoro generator. So we have

$$
\begin{aligned}
T \equiv T_+ + t_+ &= \frac{1}{4}\Pi^m\Pi_m + \partial_\sigma\Theta^\alpha\tilde{D}_\alpha\,, &(3.12)\\
\{T(\sigma),\, T(\sigma')\}_P &= 2T(\sigma)\delta'(\sigma-\sigma') + \partial_\sigma T(\sigma)\delta(\sigma-\sigma')\,. &(3.13)
\end{aligned}
$$

With respect to T, the basic quantities transform as conformal primaries. In our scheme, a conformal primary A_n of dimension n transforms as

$$\{T(\sigma),\, A_n(\sigma')\}_P = nA_n(\sigma)\delta'(\sigma-\sigma') + (n-1)\partial_\sigma A_n(\sigma)\delta(\sigma-\sigma')\,. \tag{3.14}$$

This is somewhat different from the usual form, but is equivalent to it. The familiar form arises if we expand around σ' instead of σ (and regard $-T$ as the generator), but the form above is computationally more convenient. One can easily check that a product of conformal primaries is again a conformal primary, using the formula above. Θ^α is a primary of dimension 0, while $\Pi^m, \tilde{D}_\alpha, D_\alpha, \partial_\sigma\Theta^\alpha$ are primaries of dimension 1.

Now focus on the relation (3.9), i.e. $\{\tilde{D}_\alpha(\sigma),\, \tilde{D}_\beta(\sigma')\}_P = 2i\gamma^m_{\alpha\beta}\Pi_m\delta(\sigma-\sigma')$. As $\Pi^m\Pi_m$ is a constraint, we have the familiar situation that a half of $\tilde{D}_\alpha$ is of second class and the other half is of first class. To separate them we shall use the light-cone decomposition. Although it breaks the Lorentz covariance, it does so only for terms involving $\tilde{\theta}^\alpha$ and does not affect θ^α. This feature will be one of the keys for producing the Lorentz covariant PS formalism in terms of θ^α in the end. We split a 10-dimensional chiral spinor ψ^α into SO(8) chiral and anti-chiral components as ψ^a and $\psi^{\dot{a}}$ respectively and adopt the light-cone conventions such as $\gamma^\pm \equiv \gamma^0 \pm \gamma^9, \Pi^\pm = \Pi^0 \pm \Pi^9$, etc. The transverse components of a vector A^m will be denoted as[4] A^i with $i = 1 \sim 8$. It will be useful to remember that

$$
\begin{aligned}
&\gamma^+_{\dot{a}\dot{b}} = -2\delta_{\dot{a}\dot{b}}\,, \qquad (\gamma^+)^{ab} = 2\delta^{ab}\,, \qquad \gamma^-_{ab} = -2\delta_{ab}\,, \qquad (\gamma^-)^{\dot{a}\dot{b}} = 2\delta^{\dot{a}\dot{b}}\,, \qquad \text{rest} = 0\,,\\
&\gamma^i_{a\dot{b}}\gamma^i_{c\dot{d}} + \gamma^i_{a\dot{d}}\gamma^i_{c\dot{b}} = 2\delta_{ac}\delta_{\dot{b}\dot{d}}\,. \qquad (3.15)
\end{aligned}
$$

[4]Although the index i was previously used for the the worldsheet vector index, there should be no confusion.

The Poisson bracket for the SO(8) chiral components of $\tilde{D}_\alpha$ then reads

$$\{\tilde{D}_a(\sigma),\, \tilde{D}_b(\sigma')\}_P = 2i\delta_{ab}\Pi^+\delta(\sigma-\sigma')\,. \tag{3.16}$$

This shows that $\tilde{D}_a$'s are the second class constraints, since, as is customary, we will assume that Π^+ does not vanish. As for $\tilde{D}_{\dot{a}}$, we will replace it by the combination

$$\tilde{K}_{\dot{a}} \equiv \tilde{D}_{\dot{a}} - \frac{1}{\Pi^+}\Pi^i\gamma^i_{\dot{a}b}\tilde{D}_b\,. \tag{3.17}$$

This essentially generates the κ-transformations. The bracket of $\tilde{K}_{\dot{a}}$ with $\tilde{D}_b$ is given by

$$\{\tilde{K}_{\dot{a}}(\sigma),\, \tilde{D}_b(\sigma')\}_P = 4i\gamma^i_{\dot{a}c}\gamma^i_{b\dot{d}}\frac{\partial_\sigma\Theta_{\dot{d}}\tilde{D}_c}{\Pi^+}\delta(\sigma-\sigma')\,, \tag{3.18}$$

which is proportional to the constraint $\tilde{D}_c$. The bracket of $\tilde{K}_a$ with itself is somewhat more involved and takes the form

$$\{\tilde{K}_{\dot{a}}(\sigma),\, \tilde{K}_{\dot{b}}(\sigma')\}_P = -8i\delta_{\dot{a}\dot{b}}(\mathcal{T}+\mathcal{K})(\sigma)\delta(\sigma-\sigma') + \tilde{D}\ \text{term}\,, \tag{3.19}$$

where "$\tilde{D}$ term" signifies a term proportional to $\tilde{D}_a$ and

$$\mathcal{T} \equiv \frac{T}{\Pi^+}\,, \qquad \mathcal{K} \equiv \frac{1}{\Pi^+}\tilde{K}_{\dot{c}}\partial_\sigma\Theta_{\dot{c}}\,. \tag{3.20}$$

The new operators $\mathcal{T}$ and $\mathcal{K}$, which are proportional to the constraints, enjoy the following properties:

$$\{\mathcal{T}(\sigma),\, \mathcal{T}(\sigma')\}_P = 0\,, \tag{3.21}$$

$$\{\mathcal{T}(\sigma),\, \mathcal{K}(\sigma')\}_P = \frac{\mathcal{K}}{\Pi^+}(\sigma)\delta'(\sigma-\sigma')\,, \qquad \{\mathcal{T}(\sigma),\, \tilde{D}_a(\sigma')\}_P = \frac{\tilde{D}_a}{\Pi^+}(\sigma)\delta'(\sigma-\sigma')\,, \tag{3.22}$$

$$\{\mathcal{K}(\sigma),\, \mathcal{K}(\sigma')\}_P = -2\frac{\mathcal{K}}{\Pi^+}(\sigma)\delta'(\sigma-\sigma') - \partial\left(\frac{\mathcal{K}}{\Pi^+}\right)(\sigma)\delta(\sigma-\sigma') + \tilde{D}\ \text{terms}\,, \tag{3.23}$$

$$\{\mathcal{K}(\sigma),\, \tilde{D}_a(\sigma')\}_P = -\frac{4i}{\Pi^{+2}}\gamma^i_{a\dot{b}}\gamma^i_{c\dot{d}}\tilde{D}_c\partial_\sigma\Theta_{\dot{d}}\partial_\sigma\Theta_{\dot{b}}(\sigma)\delta(\sigma-\sigma')\,. \tag{3.24}$$

It is also clear that their brackets with the original constraints, $\tilde{K}_{\dot{a}}$ and T, again close into constraints. This shows that they are completely of first class. Note especially that $\mathcal{T} = T/\Pi^+$ commutes with itself because Π^+ is a primary field of dimension 1 with respect to T.

We may now eliminate the second class constraint $\tilde{D}_a$ by employing the Dirac bracket $\{A(\sigma), B(\sigma')\}_D = \{A(\sigma), B(\sigma')\}_P - \int d\sigma_1 d\sigma_2 \{A(\sigma), \tilde{D}_a(\sigma_1)\}_P C^{ab}(\sigma_1,\sigma_2)\{\tilde{D}_b(\sigma_2), B(\sigma')\}_P$, where C^{ab} is given, from (3.16), by $C^{ab}(\sigma_1,\sigma_2) = (1/2i\Pi^+)\delta^{ab}\delta(\sigma_1-\sigma_2)$, and we may set $\tilde{D}_a = 0$ strongly after computing the bracket. It is easy to see that the effect is simply to set $\tilde{D}_a$'s to zero in the Poisson bracket relations shown above.

To further simplify the system, we will fix the constraint $\tilde{K}_{\dot{a}}$ (and hence $\mathcal{K}$) by choosing the so-called "semi-light-cone" gauge [17] for $\tilde{\theta}$. Namely, we will impose the condition $\gamma^+\tilde{\theta} = 0$, or equivalently $\tilde{\theta}_{\dot{a}} = 0$. Note that this gauge choice does not break the global spacetime supersymmetry as defined in (2.3) and (2.4) since $\tilde{\theta}$ is invariant. One must now make a further modification of the Dirac bracket, to be denoted by $\{A,\, B\}_{D^*}$, due to this gauge-fixing. Writing $\phi_I = (\tilde{\theta}_{\dot{a}}, \tilde{K}_{\dot{a}})$, it is given by

$$\begin{aligned}\{A(\sigma),\, B(\sigma')\}_{D^*} &= \{A(\sigma),\, B(\sigma')\}_D - \\ &\quad - \int d\sigma_1 d\sigma_2 \{A(\sigma),\, \phi_I(\sigma_1)\}_D C^{IJ}(\sigma_1, \sigma_2)\{\phi_J(\sigma_2),\, B(\sigma')\}_D\,,\end{aligned} \tag{3.25}$$

where the matrix C^{IJ}, the inverse of $\{\phi_I,\, \phi_J\}_D$, after setting $\phi_I = 0$ takes the form

$$C^{IJ}(\sigma_1, \sigma_2)\big|_{\phi_I=0} = \delta_{\dot{a}\dot{b}}\delta(\sigma_1 - \sigma_2)\begin{pmatrix} 8i\mathcal{T} & -1 \\ -1 & 0 \end{pmatrix}. \tag{3.26}$$

From this and the previous formulas, it is easy to see that all the brackets between the constraints vanish except for $\{\Delta_{\dot{a}},\, \Delta_{\dot{\beta}}\}_{D^*}$, which equals $\{D_{\dot{a}},\, D_{\dot{\beta}}\}_{D^*}$ upon setting all the second class constraints to zero. In this way, we obtain the following strikingly simple first class algebra which governs the entire classical dynamics of the theory:

$$\{D_{\dot{a}}(\sigma),\, D_{\dot{\beta}}(\sigma')\}_{D^*} = -8i\mathcal{T}(\sigma)\delta_{\dot{a}\dot{b}}\delta(\sigma - \sigma')\,, \tag{3.27}$$

$$\{D_a(\sigma),\, D_{\dot{b}}(\sigma')\}_{D^*} = \{D_a(\sigma),\, D_b(\sigma')\}_{D^*} = 0\,, \tag{3.28}$$

$$\{\mathcal{T}(\sigma),\, D_a(\sigma')\}_{D^*} = \{\mathcal{T}(\sigma),\, D_{\dot{a}}(\sigma')\}_{D^*} = \{\mathcal{T}(\sigma),\, \mathcal{T}(\sigma')\}_{D^*} = 0\,. \tag{3.29}$$

Also, the explicit forms of D_α and $\mathcal{T}$ are significantly simplified in the semi-lightcone gauge.

Although we have been able to simplify the structure of the theory considerably, there seems to be a large price to pay: Due to the use of the Dirac bracket, brackets between the fundamental variables are no longer canonical. For example, one finds $\{x^m(\sigma),\, k^n(\sigma')\}_{D^*} = \eta^{mn}\delta(\sigma-\sigma') + (i/2\Pi^+)(\gamma^m\tilde{\theta})_a(\gamma^n\Theta)_a(\sigma)\delta'(\sigma-\sigma')$, $\{k^m(\sigma),\, k^n(\sigma')\}_{D^*} = -(i/2)\partial_\sigma[(1/\Pi^+)(\gamma^m\Theta)_a(\gamma^n\Theta)_a\delta'(\sigma-\sigma')]$, and so on. This is natural as the original action is highly non-linear but it is disastrous especially for quantization.

4. Free field basis

Remarkably, there is a gratifying solution: One can find judicious redefinitions of the fundamental fields in such a way that the Dirac brackets among the new variables are exactly canonical. In other words, there is a basis in which all the fields are free. Below we summarize our results, now including the "right" as well as the "left" sector. The string coordinate x^m is unchanged. The new momenta p^m, p_a^A and $p_{\dot{a}}^A$ are given by

$$p^m \equiv k^m - i\partial_\sigma(\tilde{\theta}\gamma^m\theta) + i\partial_\sigma(\hat{\tilde{\theta}}\gamma^m\hat{\theta})\,, \tag{4.1}$$

$$\begin{aligned}p_a^A &\equiv k_a^A - i\eta_A(\partial_\sigma x^+ - i\theta^A\gamma^+\partial_\sigma\theta^A)\tilde{\theta}_a^A + \\ &\quad + \eta_A\left[2(\gamma^i\partial_\sigma\theta^A)_a\tilde{\theta}^A\gamma^i\theta^A + (\gamma^i\theta^A)_a\partial_\sigma(\tilde{\theta}^A\gamma^i\theta^A)\right],\end{aligned} \tag{4.2}$$

$$\begin{aligned}p_{\dot{a}}^A &\equiv k_{\dot{a}}^A + i\eta_A(\gamma^m\theta^A)_{\dot{a}}[-2i\tilde{\theta}^A\gamma_m\partial_\sigma\theta^A + i\tilde{\theta}^A\gamma_m\partial_\sigma\tilde{\theta}^A - i\partial_\sigma(\tilde{\theta}^A\gamma_m\theta^A)] - \\ &\quad - i\eta_A(\gamma^i\tilde{\theta}^A)_{\dot{a}}\left[\partial_\sigma x^i - 3i\theta^A\gamma^i\partial_\sigma\theta^A + 2i\theta^A\gamma^i\partial_\sigma\tilde{\theta}^A + i\partial_\sigma(\tilde{\theta}^{A'}\gamma^i\theta^{A'})\right],\end{aligned} \tag{4.3}$$

where the index A' for the last term of $p^A_{\dot{a}}$ signifies the one opposite to A, namely $A' = 1(2)$ if $A = 2(1)$. As for $\tilde{\theta}^A$, we actually need to regard the combinations

$$S_a = \sqrt{2\Pi^+}\,\tilde{\theta}_a\,, \quad \hat{S}_a = \sqrt{2\hat{\Pi}^+}\,\hat{\tilde{\theta}}_a\,, \tag{4.4}$$

as our fundamental fields. With these redefinitions it is straightforward to verify the following canonical Dirac bracket relations:

$$\{x^m(\sigma),\, p^n(\sigma')\}_{D^*} = \eta^{mn}\delta(\sigma-\sigma')\,, \qquad \{\theta^{A\alpha}(\sigma),\, p^B_\beta(\sigma')\}_{D^*} = -\delta^{AB}\delta^\alpha_\beta\delta(\sigma-\sigma')\,, \tag{4.5}$$

$$\{S^A_a(\sigma),\, S^B_b(\sigma')\}_{D^*} = i\delta^{AB}\delta_{ab}\delta(\sigma-\sigma')\,, \qquad \text{rest} = 0\,. \tag{4.6}$$

Another non-trivial and satisfying feature of the above redefinitions is that in terms of the new fields *complete separation* of the left and right sectors takes place: Not only does the algebra of constraints close separately in each sector (as has already been the case even before the redefinitions), the constraints in the left (right) sector are now expressed solely in terms of the left- (right-) variables. For instance, the form of Π^m, which is a building block of $\mathcal{T}$, changes as

$$\begin{aligned}\Pi^m &= k^m + \eta[\partial_\sigma x^m - 2i\theta\gamma^m\partial_\sigma\theta - 2i\tilde{\theta}\gamma^m\partial_\sigma\tilde{\theta} + 4i\tilde{\theta}\gamma^m\partial_\sigma\theta] - i\partial_\sigma(\tilde{\theta}\gamma^m\theta) + i\partial_\sigma(\hat{\tilde{\theta}}\gamma^m\hat{\theta}) \\ &= p^m + \eta[\partial_\sigma x^m - 2i\theta\gamma^m\partial_\sigma\theta - 2i\tilde{\theta}\gamma^m\partial_\sigma\tilde{\theta} + 4i\tilde{\theta}\gamma^m\partial_\sigma\theta]\,,\end{aligned} \tag{4.7}$$

where we used the "left-favored" notation that for the left sector $\eta = +1$ and the fields are as shown, while for the right sector (i.e. for $\hat{\Pi}^m$) $\eta = -1$ and we should put hats on θ and $\tilde{\theta}$ for the part within the parenthesis [ ]. This convenient notation will be used for the rest of this paper. Evidently the shift from k^m to p^m removes the variables of the "wrong sector".

Now let us display the explicit form of the constraints in terms of the canonical free fields, obtained by using (4.1) $\sim$ (4.4). Employing the "left-favored" notation introduced above, the results are

$$D_a = d_a + i\sqrt{2\Pi^+}\,S_a\,, \tag{4.8}$$

$$D_{\dot{a}} = d_{\dot{a}} + i\sqrt{\frac{2}{\Pi^+}}\Pi^i(\gamma^i S)_{\dot{a}} + \frac{2\eta}{\Pi^+}(\gamma^i S)_{\dot{a}}(S\gamma^i\partial_\sigma\theta)\,, \tag{4.9}$$

$$\mathcal{T} = \frac{1}{4}\frac{\Pi^m\Pi_m}{\Pi^+}\,. \tag{4.10}$$

Here Π^m is given in (4.7) and d_a and $d_{\dot{a}}$ are the SO(8) components of the covariant spinor d_α defined by

$$d_\alpha \equiv p_\alpha - i(\gamma^m\theta)_\alpha(p_m + \eta\partial x_m) - \eta(\gamma^m\theta)_\alpha(\theta\gamma_m\partial\theta)\,. \tag{4.11}$$

Remarkably, the 17 first class constraints (4.8) $\sim$ (4.10) will be seen to be identical, upon quantization, to the ones constructed in [16].

5. Quantization and derivation of PS formalism

Now that we have expressed all the constraints in terms of free fields, the quantization of the basic variables is essentially trivial. First, replacement of the Dirac brackets by the quantum brackets yields $[p^m(\sigma), x^n(\sigma')] = -i\eta^{mn}\delta(\sigma-\sigma')$, $\{p^A_\alpha(\sigma), \theta^{A\beta}(\sigma')\} = -i\delta^\beta_\alpha\delta(\sigma-\sigma')$ and

$\{S_a(\sigma), S_b(\sigma')\} = -\delta_{ab}\delta(\sigma - \sigma')$. Next we translate them to the OPE's in the Euclidean formulation. To obtain the standard normalization, we reinstate the string tension $T = 1/(2\pi\alpha') = 1/(4\pi)$ with the choice $\alpha' = 2$, Euclideanize, make a conformal transformation to the plane coordinate, and make the identification and redefinition[5] of the form (focussing for simplicity on the left sector) $p^m + (1/4\pi)\partial_\sigma x^m \to (i/2\pi)\partial x^m$, $p_\alpha \to p_\alpha/(2\pi i)$, and $S_a \to -iS_a/\sqrt{2\pi}$, where $\partial \equiv \partial_z$. Then the OPE's for the basic variables become

$$x^m(z)x^n(w) = -\eta^{mn}\ln(z-w)\,, \qquad p_\alpha(z)\theta^\beta(w) = \frac{\delta_\alpha^\beta}{z-w}\,, \qquad S_a(z)S_b(w) = \frac{\delta_{ab}}{z-w}\,. \tag{5.1}$$

Correspondingly, it is convenient to make the following replacements

$$D_\alpha \to \frac{1}{2\pi i}D_\alpha\,, \qquad d_\alpha \to \frac{1}{2\pi i}d_\alpha\,, \qquad \Pi^m \to \frac{1}{2\pi}\Pi^m\,, \tag{5.2}$$

and afterwards rescale $\mathcal{T}$ so that $\mathcal{T} \equiv (1/2)\Pi^m\Pi_m/\Pi^+$. Further, to facilitate the comparison with the result of [16], we will make explicit the dependence on S_a by introducing a quantity π^m defined by

$$\pi^m \equiv i\partial x^m + \theta\gamma^m\partial\theta\,. \tag{5.3}$$

This is nothing but the S_a-independent part of Π^m, which after the Euclideanization and rescaling reads

$$\Pi^m = \pi^m + i\left(\frac{i}{2\pi^+}S\gamma^m\partial S - \sqrt{\frac{2}{\pi^+}}S\gamma^m\partial\theta\right)\,. \tag{5.4}$$

Note that $\Pi^+ = \pi^+$ holds. Then, the redefined constraints in terms of the quantized fields take the form

$$\begin{aligned}
d_\alpha &= p_\alpha + i\partial x^m(\gamma_m\theta)_\alpha + \frac{1}{2}(\gamma^m\theta)_\alpha(\theta\gamma_m\partial\theta)\,, && (5.5)\\
D_a &= d_a + i\sqrt{2\pi^+}S_a\,, && (5.6)\\
D_{\dot a} &= d_{\dot a} + i\sqrt{\frac{2}{\pi^+}}\pi^i(\gamma^i S)_{\dot a} - \frac{1}{\pi^+}(\gamma^i S)_{\dot a}(S\gamma^i\partial\theta)\,, && (5.7)\\
\mathcal{T} &= \frac{1}{2}\frac{\pi^m\pi_m}{\pi^+} - \frac{1}{2\pi^+}S_c\partial S_c + i\sqrt{\frac{2}{\pi^+}}S_c\partial\theta_c + i\frac{\sqrt{2}}{(\pi^+)^{3/2}}\pi^i(S\gamma^i\partial\theta) - \frac{1}{(\pi^+)^2}(S\gamma^i\partial\theta)^2\,. && (5.8)
\end{aligned}$$

These are as yet the naive classical expressions written in terms of quantum fields. As it commonly happens, we need to add a few improvement terms in order to realize the local symmetry quantum mechanically. The necessary modifications must be related to the normal-ordering ambiguities and should cancel the double and higher poles arising from the multiple contractions which are absent in the classical computations. As for D_α there is only one term which requires normal ordering, namely the term $(-1/\pi^+)(\gamma^i S)_{\dot a}(S\gamma^i\partial\theta)$ in $D_{\dot a}$. Therefore we expect that the terms we may need are of the type $\partial^2\theta_{\dot a}/\pi^+$ and $\partial\theta_{\dot a}\partial(1/\pi^+)$. Indeed by adjusting their coefficients properly, the double and the triple poles in $D_{\dot a}(z)D_{\dot b}(w)$ can be cancelled exactly. In this way the complete quantum constraints are

[5] Appropriate removal of the factor of z^h produced for an operator of dimension h by the conformal transformation is implicitly understood.

obtained by the modifications

$$D_a \to D_a\,, \tag{5.9}$$

$$D_{\dot a} \to D_{\dot a} + \frac{4\partial^2\theta_{\dot a}}{\pi^+} - \frac{2\partial\pi^+\partial\theta_{\dot a}}{(\pi^+)^2}\,, \tag{5.10}$$

$$\mathcal{T} \to \mathcal{T} + \frac{4\partial^2\theta_{\dot c}\partial\theta_{\dot c}}{(\pi^+)^2} - \frac{1}{2}\frac{\partial^2 \ln \pi^+}{\pi^+}\,. \tag{5.11}$$

Now they close under the OPE as

$$D_{\dot a}(z)D_{\dot b}(w) = \frac{-4\delta_{\dot a\dot b}\mathcal{T}(w)}{z-w}\,, \qquad \text{other OPE's} = \text{regular}\,. \tag{5.12}$$

Up to some difference in conventions, this result agrees precisely with the one constructed in [16] by adding free fields $(p_\alpha, \theta^\alpha)$ to the GS formalism in the semi-lightcone gauge.[6] In the present formalism, we have been able to derive it rather straightforwardly from the fundamental action together with the justification of the use of free fields, which has hitherto been a postulate.

The rest of the procedure to get to the PS formalism was already fully explained in [16]. However, for completeness and for the convenience of the reader, we shall briefly reproduce the argument (restricting to the holomorphic sector) in a slightly more streamlined fashion below.

First from the simple structure of the constraint algebra, one can immediately construct, in a completely conventional way, the nilpotent BRST charge $\tilde{Q}$ in the form

$$\tilde{Q} = \int \frac{dz}{2\pi i}\left(\tilde\lambda^\alpha D_\alpha + \mathcal{T}c - (\tilde\lambda\gamma^+\tilde\lambda)b\right)\,, \tag{5.13}$$

where $\tilde\lambda^\alpha$ is an *unconstrained* bosonic spinor ghost and (b,c) is a canonical pair of fermionic ghosts satisfying $b(z)c(w) = 1/(z-w)$. At this stage the theory is defined in an extended Hilbert space similarly to the formulation in [11]. Due to this feature, one can construct the "B-ghost" and express the energy-momentum tensor as

$$B = b\pi^+ - \tilde\omega_\alpha\partial\theta^\alpha\,, \tag{5.14}$$

$$T = \left\{\tilde{Q},\, B\right\} = \frac{1}{2}\pi^m\pi_m - d_\alpha\partial\theta^\alpha - \tilde\omega_\alpha\partial\tilde\lambda^\alpha - b\partial c - \frac{1}{2}\partial^2\ln\pi^+\,, \tag{5.15}$$

where $\tilde\omega_\alpha$ is the field conjugate to $\tilde\lambda^\alpha$ satisfying $\tilde\lambda^\alpha(z)\tilde\omega_\beta(w) = \delta^\alpha_\beta/(z-w)$.

The next step is to show that the cohomology of $\tilde{Q}$ is the same as that of

$$Q^{(1)} = \int \frac{dz}{2\pi i}\left(\tilde\lambda_a D_a + \lambda_{\dot a} D_{\dot a}\right)\,, \quad \lambda_{\dot a}\lambda_{\dot a} = 0\,, \tag{5.16}$$

obtained from $\tilde{Q}$ by dropping the terms containing (b,c) and imposing a constraint $\lambda_{\dot a}\lambda_{\dot a} = 0$ or $\lambda\gamma^+\lambda = 0$. (We remove tilde to indicate that it is constrained.) Note that this is one of the five independent constraints expressed by the pure spinor conditions $\lambda\gamma^m\lambda = 0$. One

[6] The idea of adding extra free spinors to construct a first class algebra appeared earlier for the $d = 10$ [9] and $d = 11$ superparticle [21].

way to do this is to employ the logic of the homological perturbation theory [22], which is essentially the somewhat lengthy analysis presented in [16]. A more direct method is to connect $\tilde{Q}$ and $Q^{(1)}$ by the following similarity transformation, which can be easily checked:

$$e^X \tilde{Q} e^{-X} = \delta_b + Q^{(1)} , \tag{5.17}$$

$$\delta_b = 2 \int \frac{dz}{2\pi i} \tilde{\lambda}_{\dot{b}} \tilde{\lambda}_{\dot{b}} b , \qquad X = \int \frac{dz}{8\pi i} c(l_{\dot{a}} D_{\dot{a}}) . \tag{5.18}$$

Here $l_{\dot{a}}$ is an auxiliary SO(8) anti-chiral spinor with the property $\tilde{\lambda}_{\dot{a}} l_{\dot{a}} = 1$, $l_{\dot{a}} l_{\dot{a}} = 0$ and $\lambda_{\dot{a}}$ in $Q^{(1)}$ is given by $\lambda_{\dot{a}} = \tilde{\lambda}_{\dot{a}} - \frac{1}{2}(\tilde{\lambda}_{\dot{b}} \tilde{\lambda}_{\dot{b}}) l_{\dot{a}}$. This indeed satisfies $\lambda_{\dot{a}} \lambda_{\dot{a}} = 0$ and we may hereafter forget about $l_{\dot{a}}$ as it appears only in $\lambda_{\dot{a}}$. Since δ_b is completely independent of $Q^{(1)}$ and has a trivial cohomology, we may drop it to obtain $Q^{(1)}$.

The final process is to cohomologically decouple S_a together with four more degrees of freedom of $\tilde{\lambda}^\alpha$ by a judicious similarity transformation. To this end, define the following projection operators P^1 and P^2 in the SO(8)-chiral space:

$$\delta_{ab} = P^1_{ab} + P^2_{ab} , \qquad P^1_{ab} \equiv \frac{1}{2} (\gamma^i \lambda)_a (\gamma^i r)_b \equiv P^2_{ba} , \tag{5.19}$$

where again an anti-chiral spinor $r_{\dot{a}}$ with the properties $\tilde{\lambda}_{\dot{a}} r_{\dot{a}} = 1$, $r_{\dot{a}} r_{\dot{a}} = 0$ has been introduced.[7] Using these projection operators, one can decompose the self-conjugate field S_a into a "conjugate pair" (S^1_a, S^2_a) as $S^I_a = P^I_{ab} S_b, (I = 1, 2)$. They satisfy the OPE's $S^1_a(z) S^1_b(w) = S^2_a(z) S^2_b(w) = \text{(regular)}$, $S^1_a(z) S^2_b(w) = P^1_{ab}/(z-w)$ and $S^2_a(z) S^1_b(w) = P^2_{ab}/(z-w)$. Similarly, $\tilde{\lambda}_a$ is decomposed into $\tilde{\lambda}_a = \lambda^1_a + \lambda^2_a$, where $\lambda^I_a = P^I_{ab} \tilde{\lambda}_b$. It is important to note that λ^1_a is easily checked to satisfy

$$\lambda^1_a \gamma^i_{a\dot{b}} \lambda_{\dot{b}} = 0 , \tag{5.20}$$

which are the remaining four independent equations contained in the pure spinor constraints.

The similarity transformation is best performed in two steps.[8] First, we make a transformation $Q^{(2)} = e^Y Q^{(1)} e^{-Y}$ with

$$Y = -\frac{1}{2} \int \frac{dz}{2\pi i} S^1_a S^2_a \ln \pi^+ . \tag{5.21}$$

The main effect of this transformation is the replacement

$$S^1_a \to \frac{S^1_a}{\sqrt{\pi^+}} , \qquad S^2_a \to \sqrt{\pi^+}\, S^2_a , \tag{5.22}$$

which shifts the conformal weight of (S^1_a, S^2_a) from $(1/2, 1/2)$ to $(1, 0)$, the latter being more natural for a conjugate pair. Including the remaining effects, $Q^{(2)}$ becomes (omitting

[7] As remarked in [16], dimension 0 operator $\theta_{\dot{a}} r_{\dot{a}}$ containing $r_{\dot{a}}$ should be excluded from the Hilbert space to avoid the triviality of the cohomology.

[8] We reverse the order of the two transformations compared to [16] for added clarity.

integral signs for simplicity)

$$
\begin{aligned}
Q^{(2)} &= \delta + Q + d\,, && (5.23)\\
\delta &= \sqrt{2}\,i\lambda^2_a S^1_a\,, \qquad Q = \lambda^1_a d_a + \lambda_{\dot a} d_{\dot a}\,, && (5.24)\\
d &= \frac{4\lambda_{\dot a}\partial^2\theta_{\dot a}}{\pi^+} - \frac{4\partial\pi^+\lambda_{\dot a}\partial\theta_{\dot a}}{(\pi^+)^2} + \lambda^2_a d_a + \\
&\quad + \sqrt{2}\,i(\pi^+\lambda^1_a S^2_a + \pi^i(\lambda\gamma^i S^2)) - (\lambda\gamma^i S^2)(S^2\gamma^i\partial\theta)\,, && (5.25)
\end{aligned}
$$

where the first two terms in d come from truly quantum contributions. Note that if we assign the degrees $\deg(S^1_a, S^2_a) = (-1, +1)$, d includes all the terms of positive degree while δ and Q carry degrees -1 and 0 respectively. With this grading structure in mind, it is an easy matter to find a similarity transformation which removes d entirely:

$$
\begin{aligned}
Q^{(3)} &= e^Z Q^{(2)} e^{-Z} = \delta + Q\,, && (5.26)\\
Z &= \int \frac{dz}{2\pi i}\left(-\frac{d_a S^2_a}{\sqrt{2}\,i} + \frac{4(\partial\theta_{\dot a}\lambda_{\dot a})(\partial\theta_{\dot b} r_{\dot b})}{\pi^+}\right). && (5.27)
\end{aligned}
$$

Now since δ is completely independent of Q and its cohomology is easily seen to be trivial, we may drop δ as well. Finally, by renaming $\lambda^1_a \to \lambda_a$ and denoting $\lambda^\alpha = (\lambda_a, \lambda_{\dot a})$, we reach the simple BRST operator Q of the PS formalism:

$$
\begin{aligned}
Q &= \int \frac{dz}{2\pi i}\lambda^\alpha d_\alpha\,, \qquad \lambda\gamma^m\lambda = 0\,, && (5.28)\\
d_\alpha &= p_\alpha + i\partial x^m(\gamma_m\theta)_\alpha + \frac{1}{2}(\gamma^m\theta)_\alpha(\theta\gamma_m\partial\theta)\,. && (5.29)
\end{aligned}
$$

6. Application to superparticle in 11 dimensions

Evidently, our formalism described above for the superstring contains, as its zero mode sector, the case of a superparticle in 10 dimensions. It is in fact much simpler than for the superstring, because the bulk of the non-trivial features of the string case, such as the necessity of the redefinitions to get free fields etc., are due to expressions involving σ-derivatives, and they are absent for a particle.

As we will now briefly show, our basic idea works almost verbatim for a superparticle in 11 dimensions as well: Starting from a covariant Brink-Schwarz type action one can straightforwardly derive the PS formalism[9] for it, which coincides with the one introduced in [20].

Let us first summarize the conventions and properties of the Γ-matrices and spinors to be used. 32×32 real Γ-matrices will be denoted by Γ^M_{AB}, ($M = 0 \sim 10$; $A, B = 1 \sim 32$), the charge conjugation matrix $C = \Gamma^0$ has the property $C^T = -C$ and $C\Gamma^M$ and $\Gamma^M C$ are symmetric. The lightcone decomposition of Γ^M is taken as $\Gamma^\pm = \Gamma^0 \pm \Gamma^{10}$, Γ^i ($i = 1 \sim 9$). The lightcone chirality operator (i.e. SO(1,1) boost charge) is defined as $\hat\Gamma \equiv \Gamma^0\Gamma^{10}$ and a 32-component spinor χ_A will be decomposed according to the eigenvalues of $\hat\Gamma$ as

[9]Strictly speaking, the notion of the pure spinor in 11 dimensions in the sense of Cartan [23] requires additional conditions, but we will continue to refer to it as PS formalism.

JHEP05(2005)046

$\chi_A = (\chi_\alpha, \chi_{\dot\alpha})$, $(\alpha, \dot\alpha = 1 \sim 16)$, with $\hat\Gamma_{\alpha\beta}\chi_\beta = +\chi_\alpha$, $\hat\Gamma_{\dot\alpha\dot\beta}\chi_{\dot\beta} = -\chi_{\dot\alpha}$. Note that this decomposition is in parallel with the SO(8) decomposition for the 10-dimensional case and differs from the one with respect to the 10-dimensional chirality operator Γ^{10}.

We start from the covariant action of the form

$$S = \int dt L\,, \qquad L = \frac{1}{2e}\Pi^M \Pi_M\,, \tag{6.1}$$

where e is the einbein and Π^M is given by

$$\Pi^M = \dot x^M - i\partial_t(\bar\theta\Gamma^M\tilde\theta) + i\dot{\bar\Theta}\Gamma^M\Theta\,, \qquad \Theta \equiv \tilde\theta - \theta\,, \tag{6.2}$$

where $\bar\theta$ denotes the usual Dirac conjugate θC. This action is invariant under the reparametrization, the super-Poincaré transformation, and the following three fermionic transformations: Namely, the global supersymmetry transformation

$$\delta\theta_A = \epsilon_A\,, \quad \delta\tilde\theta_A = 0\,, \quad \delta x^M = i\bar\epsilon\Gamma^M\theta\,, \tag{6.3}$$

the local supersymmetry transformation

$$\delta\theta_A = \delta\tilde\theta_A = \chi_A\,, \quad \delta x^M = i\bar\chi\Gamma^M\Theta\,, \tag{6.4}$$

and the κ-transformation, to be described below.

After the standard hamiltonian analysis and choosing the gauge $e = 1$, one finds that the total hamiltonian consists of arbitrary linear combination of the constraints of the form

$$D_A = p_A - i(C\not p(\theta - 2\tilde\theta))_A = 0\,, \tag{6.5}$$
$$\tilde D_A = \tilde p_A - i(C\not p\tilde\theta)_A = 0\,, \tag{6.6}$$
$$T = \frac{1}{2}p^2 = 0\,, \tag{6.7}$$

where p^M, p_A and $\tilde p_A$ are the momenta conjugate to x_M, θ_A and $\tilde\theta_A$ respectively and $\not p \equiv p_M\Gamma^M$. With the basic Poisson brackets $\{x^M, p^N\}_P = \eta^{MN}, \{\theta_A, p_B\}_P = -\delta_{AB}$ and $\{\tilde\theta_A, \tilde p_B\}_P = -\delta_{AB}$, they satisfy the algebra

$$\{D_A, D_B\}_P = \{\tilde D_A, \tilde D_B\} = 2i(C\not p)_{AB}\,, \qquad \{D_A, \tilde D_B\}_P = -2i(C\not p)_{AB}\,, \tag{6.8}$$
$$\{T, T\}_P = \{T, D_A\}_P = \{T, \tilde D_A\}_P = 0\,. \tag{6.9}$$

Again the combination $\Delta_A \equiv D_A + \tilde D_A$ commutes with all the constraints.

To dissociate the second class part from the the first class part, we invoke the decomposition with respect to the lightcone chirality and employ the κ-symmetry generator $\tilde K_{\dot\alpha}$ in place of $\tilde D_{\dot\alpha}$:

$$\tilde K_{\dot\alpha} \equiv \tilde D_{\dot\alpha} - \frac{1}{p^+}p^i(C\Gamma^i)_{\dot\alpha\beta}\tilde D_\beta\,. \tag{6.10}$$

Then we get the algebra

$$\{D_\alpha, D_\beta\}_P = 2ip^+\delta_{\alpha\beta}\,, \quad \{\tilde K_{\dot\alpha}, D_\beta\}_P = 0\,, \tag{6.11}$$
$$\{\tilde K_{\dot\alpha}, \tilde K_{\dot\beta}\}_P = -4i\frac{T}{p^+}\delta_{\dot\alpha\dot\beta}\,, \tag{6.12}$$

showing that $\tilde{D}_\alpha$ is of second class and $\tilde{K}_{\dot{\alpha}}$ is of first class. We now take the semi-lightcone gauge $\tilde{\theta}_{\dot{\alpha}} = 0$ to render the pair $(\tilde{K}_{\dot{\alpha}}, \tilde{\theta}_{\dot{\alpha}})$ second class and introduce the total Dirac bracket $\{\star, \star\}_{D^*}$ with respect to all the second class constraints including $\tilde{D}_\alpha$. Then, the variable $\tilde{\theta}_\alpha$, with a rescaling, becomes self-conjugate as

$$\{S_\alpha, S_\beta\}_{D^*} = i\delta_{\alpha\beta}\,, \qquad S_\alpha \equiv \sqrt{2p^+}\tilde{\theta}_\alpha\,, \tag{6.13}$$

and we are left with a completely first class constraint algebra:

$$\{D_A, D_B\}_{D^*} = 2i\frac{T}{p^+}(C\Gamma^+)_{AB}\,, \qquad \{T, D_A\}_{D^*} = 0\,, \quad \{T, T\}_{D^*} = 0\,. \tag{6.14}$$

As said before, for the particle case all the basic variables are already free and we can readily quantize the theory in the standard way. For convenience we make rescalings $S_\alpha \to -iS_\alpha$, $p_A \to -ip_A$ so that the quantized variables obey simpler (anti)commutation relations:

$$[p^m, x^n] = \frac{1}{i}\eta^{mn}\,, \quad \{p_A, \theta_B\} = \delta_{AB}\,, \quad \{S_\alpha, S_\beta\} = \delta_{\alpha\beta}\,. \tag{6.15}$$

We also make a redefinition $D_A \to -iD_A$. Then the quantum constraints take the form

$$D_A = d_A + \delta_A\,, \qquad T = \frac{1}{2}p^2\,, \tag{6.16}$$

$$d_A = p_A + (C\not{p})_{AB}\theta_B\,, \quad \delta_\alpha = i\sqrt{2p^+}S_\alpha\,, \quad \delta_{\dot{\alpha}} = i\sqrt{\frac{2}{p^+}}p^i(C\Gamma^i)_{\dot{\alpha}\beta}S_\beta\,, \tag{6.17}$$

and they form the algebra

$$\left\{D_{\dot{\alpha}}, D_{\dot{\beta}}\right\} = -4\frac{T}{p^+}\delta_{\dot{\alpha}\dot{\beta}}\,, \quad \{D_\alpha, D_\beta\} = \left\{D_\alpha, D_{\dot{\beta}}\right\} = [T, D_A] = [T, T] = 0\,. \tag{6.18}$$

The corresponding BRST operator is

$$\hat{Q} = \lambda_A D_A + \lambda_{\dot{\alpha}}\lambda_{\dot{\alpha}}b + \frac{2T}{p^+}c\,, \tag{6.19}$$

where λ_A and (b, c) respectively are the bosonic and fermionic ghosts. This is precisely of the same structure as the one for the 10 dimensional superparticle discussed in [16] and derived in this paper as a part of the superstring. Moreover, the rest of the procedure to decouple S_α together with a part of λ_A, elaborated in [16], leading to the PS formalism goes through verbatim by replacing the SO(8) γ-matrices γ^i, $i = 1 \sim 8$ with the SO(9) Γ-matrices $C\Gamma^i$, $i = 1 \sim 9$. The final result is the BRST operator $Q = \lambda^A d_A$, with the PS conditions $\lambda\Gamma^M C\lambda = 0$ (or $\lambda C\Gamma^M\lambda = 0$ if we redefine $C\lambda \to \lambda$) for $M = 0 \sim 10$. This was shown in [24, 20] to be the correct conditions to reproduce the spectrum of the supergravity in 11 dimensions as the cohomology of Q.

7. Summary and discussions

In this paper, we have constructed a reparametrization and super-Poicaré invariant world-sheet action from which one can derive the pure spinor formalism in a logically complete manner. The basic idea was to write the Green-Schwarz spinor field Θ as a difference of two independent fields $\tilde{\theta} - \theta$ and at the same time introduce appropriate interactions between them so that an extra compensating local supersymmetry is realized. By fixing the gauge for $\tilde{\theta}$ using κ-symmetry while untouching θ and retaining the new local supersymmetry, one is lead to a simple closed system of first class constraints. Moreover, we found highly non-trivial redefinitions of fields, under which all the basic variables become canonically free. This allowed us to quantize the theory in a straightforward manner with slight quantum modifications for the form of the constraints. Remarkably this set of quantized constraints agreed precisely with those engineered in [16]. Then as demonstrated in [16] one can immediately construct the BRST operator and show that its cohomology is equivalent to that in the PS formalism. We have also shown that our idea works equally well for the superparticle in 11 dimensions, with the emergence of the correct constraints for the bosonic spinor ghosts to describe the 11 dimensional supergravity.

There are many interesting further investigations one would like to perform based on the present formalism. Let us briefly discuss some of them below.

- One obvious problem is the path-integral reformulation of our idea. In particular, now that we can start form a fundamental action, it should be possible to derive the appropriate measure [1, 3] and gain deeper understanding from the first principle.
- Another intriguing project is the application to the supermembrane. The success for the superparticle in 11 dimensions is an encouraging sign but the full-fledged investigation for the supermembrane is expected to be highly non-trivial. In any case, such a study would no doubt shed a new light on the structure of the supermembrane dynamics.
- Although we believe that our fundamental action is the minimal one containing all the necessary ingredients, it need not be unique. There might be some advantage to embed it in a larger framework with further (local) symmetries, for instance a doubly supersymmetric formulation, so that added freedom to manipulate the gauge choice may produce interesting variants.

Some of these and related problems are currently under investigation and we hope to report on the results in future communications.

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